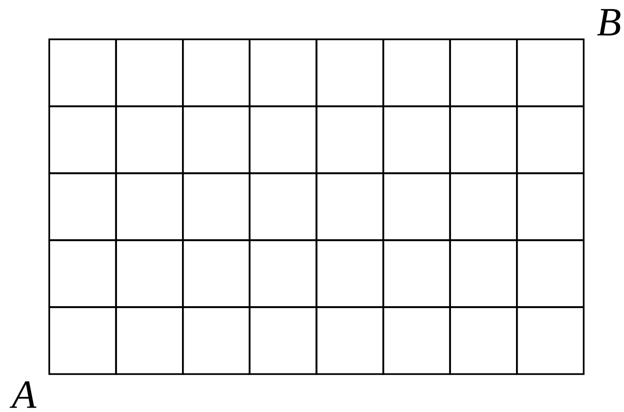
Introduction to Combinatorics

<u>The path counting problem</u>

How many paths of shortest length are there from A to B traveling along the grid?



<u>Solution 1:</u> Label each intersection with the number of paths from A to that intersection. Begin by placing 1's on all of the intersections on the leftmost vertical line, and 1's on all of the intersections on the bottommost horizontal line. (Why?) Each intersection is the sum of ____. Fill in the whole diagram.

<u>Solution 2:</u> Note that every path consists of 8 steps to the right and 5 steps up. Denote each step right by "R" and each step up by "U". Every path from A to B can be denoted by a sequence of 13 letters, 8 of which are R's and 5 of which are U's. (Consider, for example, the path designated as RRRUURUURURRR.) Moreover, any sequence of 8 R's and 5 U's spells out a unique path from A to B. So, the problem is equivalent to asking "How many unique words consist of exactly 8 R's and 5 U's?"

Suppose the R's and the U's were distinct, e.g. R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, U_1, U_2, U_3, U_4, U_5. Then there are _____ unique words using these letters. Now for any given word, there to ____ ways to rearrange just the R's, and ____ ways to rearrange just the U's. Due to this overcounting, the number of unique words when the R's and U's are indistinguishable is _________.

<u>Solution 3:</u> Consider a series of 13 blank lines

___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

We wish to place 8 R's and 5 U's in the blanks. But, once all the R's are placed we know the other blanks must be U's. How many ways are there to choose 8 of the 13 blanks? We don't care about the order in which the 8 blanks are chosen.

Lessons from the path problem

Solution 1 shows that a clever labeling method is quite useful.

Solution 2 shows that problems that involve enumeration can benefit from **transforming the problem into an equivalent problem** that is easier to solve. Then, it shows that it might be easier to **overcount at first then correct the total for the overcounting**. Finally, the solution also demonstrates the formula for letter rearrangement: (number of letters)! divided by the factorial of each repeat.

Solution 3 shows another transformation of the problem into an equivalent problem. It also shows the concept behind the very important formula for the **combination: the number of ways to choose r items from n objects, where the order in which the items are chosen doesn't matter**.

Pascal's triangle, revisited

Look at your diagram for solution 1. Tilt the diagram. Do the numbers look familiar? It's Pascal's triangle!

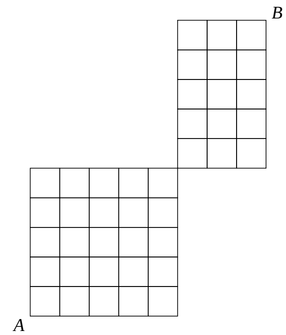
1) Why is every entry in Pascal's triangle a combination?

2) Why are combinations also coefficients in the expansion of $(x + y)^n$?

3) Why does each row of Pascal's triangle sum to a power of 2?

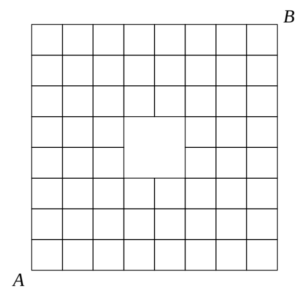
4) What is another reason each row of Pascal's triangle sums to a power of 2?

Problems



1) How many paths of shortest length are there from *A* to *B* traveling along the grid shown to the right?

2) How many 6 digit numbers have digits in strictly decreasing order? (For example, 976431 is strictly decreasing but 977432 is not.)



3) How many paths of shortest length are there from *A* to *B* traveling along the grid shown to the right?

4) Compute the number of positive factors of $1^1 2^2 3^3 4^4 5^5 6^6$.

5) How many ways can you split 10 people into 5 pairs?

6) A meeting of the Knights of the Round Table is attended by King Arthur, Sir Bedivere, Sir Caradoc, Sir Daniel, Sir Ector, and Sir Murderer. King Arthur thinks Sir Murderer will be true to his name and refuses to sit next to him. The men sit down at a circular table. How many different seating arrangements are there? (Two seating arrangements are equivalent if they are rotations of each other.)

7) Ten distinct points are placed on the circumference of a circle and chords are drawn connecting all possible pairs of these points. No three of these chords pass through the same point. Determine the number of intersection points formed inside the circle.

8) How many (true) 6 digit numbers have a sum of digits equal to 9?

Problem of the Week

How many positive integers have digits that are in strictly increasing order?

Solutions

1) [Classic]
The grid consists of two grids with a common corner. There are ${}_{10}C_5 = 252$ ways to go from A to the common corner. There are ${}_8C_3 = 56$ ways to go from the common corner to B. So, there are $252 \times 56 =$ **14112** paths in total.

2) [Classic]
Any 6 distinct digits, when ordered from greatest to least, forms a 6 digit number whose digits are in strictly decreasing order. There are ${}_{10}C_6 = 210$ ways to unique ways choose 6 of the 10 digits and hence **210** unique 6 digit numbers whose digits are in strictly decreasing order.

3) [Classic]
If the center hole were filled in, there would be ${}_{16}C_8 = 12870$ paths from A to B. Some of these paths pass through the center hole and the rest do not. Determining the number of paths passing through the center hole is just like problem #1, i.e. the grid would look like two 4×4 grids joined at a common corner. Thus, the number of paths passing through the center hole is $({}_8C_4)({}_8C_4) = (70)(70) = 4900$. So, the number of paths not passing through the center hole is $12870 - 4900 =$ **7970**.

4) [Raymond Cheong]
The prime factorization of the number is $1^1 2^2 3^3 4^4 5^5 6^6 = (2^2)(3^3)(2^8)(5^5)(2^6 3^6) = 2^{16} 3^9 5^5$. Any factor of this number has a prime factorization of $2^a 3^b 5^c$ where a is between 0 and 16, b is between 0 and 9, and c is between 0 and 5, all inclusive. Thus, there are 17 choices for a, 10 choices for b, and 6 choices for c, for a total of **1020** factors.

5) [Mathematical Circles]
Solution 1: There are ${}_{10}C_5 = 252$ ways to choose one member of each pair, then the remaining five members can be assigned to the pairs in 5! ways. But we could assign each pair in 2 different orders (A then B, or B then A) so there are 2^5 different listings. In total, the number of pairs is $252 \times 5! / 2^5 =$ **945**.

Solution 2: Label the 10 people A, B, C, D, E, F, G, H, I, and J. All pairings can be put in a "standard order" in which (1) the members in each pair are put in alphabetical order, then (2) each pair is treated as a word and the words are alphabetized. One example of 5 pairs put in this standard order is: AE, BF, CD, GJ, HI. Note that any set of 5 pairs has a unique standard order and any five 2-letter words (made of the letters A-J) in standard order represents a unique pairing. The question then becomes how many standard orderings are there?

Each standard order can be constructed by the following algorithm. First, choose the lowest alphabetical remaining person. Second, choose anyone else as the pair to this person. Then, repeat steps 1 and 2 until all the pairs are designated. By this algorithm it is easy to see there are $(1 \times 9) \times (1 \times 7) \times (1 \times 5) \times (1 \times 3) \times (1 \times 1) =$ **945** standard orderings.

Solution 3: Suppose that 2n people can be put into n pairs in M different ways. If we were to add two new people, then the number of pairings can be determined as follows. First, the two new people could be paired together and the other 2n people can be paired in the M ways. Second, one of the n pairs could be split up and paired with the two new people. There are two ways to pair with the new people, n pairs to choose from, and M different sets of pairs, for 2nM ways. So, the total number of ways to pair 2n+2 people is $M + 2nM = (2n + 1)M$.

Since 2 people (n = 1) can be put into pairs in just 1 way, we have

2 people	1 pairings
4 people	1×3 pairings
6 people	$1 \times 3 \times 5$ pairings
8 people	$1 \times 3 \times 5 \times 7$ pairings
10 people	$1 \times 3 \times 5 \times 7 \times 9 =$ **945** pairings.

6) [Inspired by AoPS, Humphrey the Magical Pigeon]
There are 6 distinct people. If the table were unfolded into a line then there would 6! total seating arrangements. Since there are 6 ways to do the unfolding, there are only 5! rotationally distinct arrangements.

Now consider the number of arrangements in which King Arthur and Sir Murderer sit next to each other. In this case, it is as if the two men were a single person, leaving us to consider the arrangement of 5 people around the table. By the logic above, there would be 4! rotationally distinct arrangements, but we need to multiply by 2 to account for the relative positions of King Arthur and Sir Murderer (Arthur sits one position clockwise or counterclockwise compared to Murderer). So the total number of arrangements where the two do NOT sit next to each other is $5! - 2(4!) =$ **72**.

7) [http://mathforum.org/library/drmath/view/51708.html, Ask Dr. Math, solution due to John Conway]
If there are n points, the number of interior intersection points is ${}_nC_4$, since the two intersecting lines that define any such point have a total of 4 ends, and any set of 4 points are the ends of just one pair of intersecting lines. So, the answer is ${}_{10}C_4 =$ **210**.

8) [Adapted from Mathematical Circles (Tom Davis, Combinatorial Problems, #24)]
Suppose for a moment that the first digit does not have to be non-zero. Then, according to the partition formula, the sum of 9 can be distributed into the 6 digits in ${}_{(9+6-1)}C_{(6-1)} = {}_{14}C_5 = 2002$ ways. (Here we don't have to worry about one of the digits exceeding 9 because the sum of digits is only 9.) However, since the first digit has to be at least one, we are only distributing a sum of 8 among the 6 digits. This can be done in ${}_{(8+6-1)}C_{(6-1)} = {}_{13}C_5 =$ **1287** ways.

Model solution to the Problem of the Week

Any set of *n* distinct non-zero digits, when ordered from least to greatest, generates a unique positive integer whose digits are in strictly increasing order. (This differs from problem #2, where we could include 0 in the set of digits.) Likewise, any positive integer whose digits are in strictly increasing order defines a unique set of *n* distinct non-zero digits. Thus, the total number of integers (adding up the number of 1-, 2-, 3-, …, 9-digit integers) is

$$ {}_9C_1 + {}_9C_2 + {}_9C_3 + \ldots + {}_9C_9 = 2^9 - {}_9C_0 = 2^9 - 1 = \boxed{\mathbf{511}}. $$

(Note that this solution properly does not count 0 as one of the numbers, because it is not positive.)