27th Annual ARML Scrimmage

Featuring:
Howard County ARML Team (host)
Baltimore County ARML Team
ARML Team Alumni
Citizens

By Raymond Cheong

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Reservoir HS

Individual Round (10 min. per pair of questions, no calculators allowed)

I-1. The sum $a + b$, the product ab, and the difference of squares $a^2 - b^2$ of two positive numbers a and b are the same number. Compute the value of b.

I-2. Let ABC be an acute triangle with altitudes $\overline{CD}$ and $\overline{AE}$, such that $BD = 3$, $AD = 5$, and $BE = 2$. Compute the length of $\overline{CE}$.

I-3. Compute all ordered pairs of positive integers (a, b) such that $a! + 4! = b^2$.

I-4. Compute the smallest positive multiple of 999 that does not have any 9's among its digits.

I-5. Compute the sum of all four-digit positive integers that are palindromes. (A number is a palindrome if it remains the same when its digits are reversed.)

I-6. The roots of $x^3 - r_1x^2 + 2012r_2x - r_3 = 0$ are r_1, r_2, and r_3, which are all non-zero. Compute the ordered triple (r_1, r_2, r_3).

I-7. In triangle ABC, $AB = 7$ and $BC = 9$. If $\text{m}\angle B = 2\text{m}\angle A$, compute the length of $\overline{AC}$.

I-8. Let f be a polynomial of degree 2010 satisfying $f(k) = 1/k$ for $k = 1, 2, \dots, 2011$. Compute the value of $f(2012)$.

I-9. Compute the number of positive integers less than 100 which have a prime number of positive divisors.

I-10. Let $a_1 = 1$, $a_2 = 1$, and $a_n = \frac{1}{20}a_{n-2} + \frac{1}{12}a_{n-1}$ for integers $n \geq 3$. Compute the value of the infinite sum $a_1 + a_2 + a_3 + \cdots$.

Team Round (20 min., no calculators allowed)

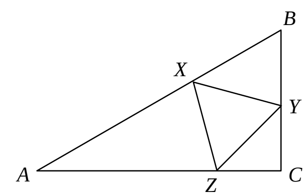
T-1. A four-digit number which is a perfect square is formed by writing Anne's age in years followed by Tom's age in years. Similarly, 31 years from now, their ages in the same order will again form a four-digit perfect square. Compute the number formed by their present ages.

T-2. Bases a and b have the property that $386_a = 272_b$ and $146_a = 102_b$. Compute the ordered pair (a, b).

T-3. Compute the number of positive integers b for which there exists an integer c, $0 \leq c \leq 2012$, such that all of the roots of $x^2 + bx + c$ and $x^2 + bx + c + 1$ are real and are integers.

T-4. Compute how many 5-digits numbers $abcde$ exist such that the digits b and d are each the sum of the digits to their immediate left and right.

T-5. Triangle ABC has $m\angle A = 30°$ and $m\angle B = 60°$. Points X, Y, and Z are on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that triangle XYZ is equilateral. If $BY = CY = 1$, compute the area of triangle XYZ.



T-6. Compute the value of $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2011^2}+\frac{1}{2012^2}}$.

T-7. The shortest side of a triangle has length 1 and the tangents of all of its angles are integers. Compute the length of the longest side of the triangle.

T-8. Compute the value of $\tan 20° \tan 40° \tan 80°$.

T-9. Compute the value of the sum $[\![\sqrt{1}]\!] + [\![\sqrt{2}]\!] + [\![\sqrt{3}]\!] + \cdots + [\![\sqrt{10000}]\!]$ where $[\![x]\!]$ denotes x rounded to the nearest integer.

T-10. Starting with a sequence of n 1's, you can insert plus signs (or no plus signs) between the 1's to get various results. For example, when $n = 3$, you can get the results 111, $11 + 1 = 12$, $1 + 11 = 12$, or $1 + 1 + 1 = 3$. Compute the number of values of n so that the result 1111 is possible.

Relay Round (6 min. per relay, no calculators allowed)

R1-1. Compute the number of positive integers n for which $((n!)!)!$ evenly divides $(2012!)!$.

R1-2. Let $T = TNYWR$. Compute the value of the infinite product $T \cdot \sqrt{T} \cdot \sqrt{\sqrt{T}} \cdot \sqrt{\sqrt{\sqrt{T}}} \cdots$.

R1-3. Let $T = TNYWR$. It is possible to place numbers in the vacant squares of the 3×3 grid shown so that the numbers along each row and column form (possibly different) arithmetic sequences. Compute the value of x.

	20	2012
	x	12
T		

R2-1. Compute the least solution to $(x-3)^{x^2+3x+2} = 1$.

R2-2. Let $T = TNYWR$ and let $f(x) = ax + b$. Compute the real value of a such that $f(f(f(1))) = 25$ and $f(f(f(0))) = T$.

R2-3. Let $T = TNYWR$. A number x is chosen at random from the interval $[0,1]$ and a number y is chosen at random from the interval $[0, T]$. Compute the probability that $x > y$.

Power Round (1 hr., no calculators allowed)

A real function f defined on the set of real numbers is called *integer-valued* if $f(x)$ is always an integer for all integers x. If f is a polynomial that has integer coefficients, then clearly it is integer-valued. But there are also polynomials with non-integer coefficients that are also integer-valued. In this round, you will find out what those polynomials are.

Definitions:

- The polynomials p_n are defined as $p_0(x) = 1$ for all x and, for all integers $n > 0$, as $p_n(x) = \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n(n-1)(n-2)\cdots(1)}$ for all x.
- For every function f define a corresponding function Δf such that $\Delta f(x) = f(x+1) - f(x)$ for all x.
- The degree of a polynomial f is the highest power of its terms with non-zero coefficients, or is zero if f is constant (including zero).

1. a. Prove that $f(x) = x(x+1)/2$ is integer-valued.
   b. Provide a simple example of a non-constant polynomial g with rational coefficients such that $g(x)$ is never an integer for all integers x.

2. Prove that $\Delta p_{n+1}(x) = p_n(x)$ for all x and all integers $n \geq 0$.

3. a. For all integers $n \geq 0$ determine (in terms of n) all real values of x such that $p_n(x) = 0$.
   b. Prove that p_n is integer-valued (i.e. $p_n(x)$ is an integer for <u>all</u> integers x) for all integers $n \geq 0$.

4. a. Prove that if f is integer-valued, then Δf is integer-valued.
   b. (*Distributive property*) Let $h(x) = uf(x) + vg(x)$ for all x for any given functions f and g and any given constants u and v. Prove that $\Delta h(x) = u\Delta f(x) + v\Delta g(x)$ for all x.

5. Prove that if f is a polynomial of degree $n \geq 1$, then Δf is a polynomial whose degree is $n-1$.

6. a. Show that if f is a polynomial and Δf is identically zero, then f is constant.
   b. Provide a simple example of a continuous, non-polynomial function g such that Δg is identically zero.

7. Show that if f and g are polynomials and $\Delta f(x) = \Delta g(x)$ for all x, then $f(x) = g(x) + C$ for all x for some constant C, and furthermore, if f and g are also integer-valued then C is an integer.

8. Let f be a polynomial of degree $n \geq 0$. Prove that f is integer-valued if and only if there exist integer coefficients $a_n, a_{n-1}, \ldots, a_0$ such that $f(x) = a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x)$ for all x.

9. (*Uniqueness*) Let f be a polynomial of degree $n \geq 0$. Prove that if, for all x, $f(x) = a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x)$ and $f(x) = b_n p_n(x) + b_{n-1}p_{n-1}(x) + \cdots + b_0 p_0(x)$ for the coefficients $a_n, a_{n-1}, \ldots, a_0$ and $b_n, b_{n-1}, \ldots, b_0$, then $a_i = b_i$ for $i = 0, 1, \ldots, n$.

10. a. Describe with proof an algorithm that, given a specific integer-valued polynomial, can determine the value of the integer coefficients (as defined in Problem 8).
    b. Compute the integer coefficients (as defined in Problem 8) for $f(x) = 2 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 2$.

Answers

Individual Round

I-1. $\left(1+\sqrt{5}\right)/2$
I-2. 10
I-3. $(1,5)$ and $(5,12)$
I-4. 111888
I-5. 495000
I-6. $(-1/2012\,,-2012,2012)$
I-7. 12
I-8. $1/1006$
I-9. 32
I-10. $115/52$

Team Round

T-1. 1225
T-2. $(9,11)$
T-3. 44
T-4. 330
T-5. $7\sqrt{3}/16$
T-6. $4048143/2012$ ($2011\frac{2011}{2012}$ is also acceptable)
T-7. $3\sqrt{5}/5$
T-8. $\sqrt{3}$
T-9. 666700
T-10. 121

Relay Round

R1-1. 6
R1-2. 36
R1-3. -478

R2-1. -2
R2-2. 3
R2-3. $1/6$

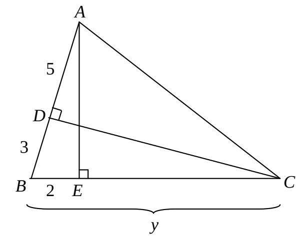
Acknowledgements

Thanks to Oleg Kryzhanovsky and George Reuter for reviewing the problems.

Individual Round Solutions

I-1. Since $a + b = a^2 - b^2 = (a + b)(a - b)$, dividing both sides by $a + b \neq 0$ yields $a - b = 1$ or $a = b + 1$. Substituting this into $ab = a + b$ yields $b(b + 1) = 2b + 1$ or $b^2 - b - 1 = 0$. The only positive root of this equation is $b = \boxed{\mathbf{(1 + \sqrt{5})/2}}$.

I-2. Solution 1: Since triangle ABC is acute, $\overline{AE}$ and $\overline{CD}$ intersect inside the triangle, as shown. Now, let $BC = y$. By the Pythagorean Theorem, $AE = \sqrt{8^2 - 2^2} = \sqrt{60}$ and $CD = \sqrt{y^2 - 9}$. So, the area of the triangle equals $\frac{1}{2}y\sqrt{60} = \frac{1}{2}8\sqrt{y^2 - 9}$. Solve as follows: $60y^2 = 64(y^2 - 9) \Rightarrow y^2 = 144 \Rightarrow y = 12$. Thus, $CE = y - 2 = \boxed{\mathbf{10}}$.



Solution 2: Since triangle ABC is acute, D lies between A and B, so $AB = AD + BD = 8$. Since right triangles CDB and AEB are similar, we conclude that $CB/AB = BD/BE$. Hence, $CB/8 = 3/2$, so $CB = 12$ and $CE = CB - BE = 12 - 2 = \boxed{\mathbf{10}}$.

Note: Can you prove that, given a triangle ABC with altitudes $\overline{CD}$ and $\overline{AE}$, such that $BD = 3$, $AD = 5$, and $BE = 2$, that ABC must be an acute triangle?

I-3. If $a \geq 6$ then $a! = 6! \cdot (2k)$ for some even integer $2k$. Consequently, $a! + 4! = 4!\,(2k + 1) = 8 \cdot 3 \cdot (2k + 1)$ which cannot be a perfect square because it is an odd multiple of 8. For $a \leq 5$, we have $1! + 4! = 25$, $2! + 4! = 26$, $3! + 4! = 30$, $4! + 4! + 48$, and $5! + 4! = 144$. Thus, the only solutions are $\boxed{\mathbf{(1, 5)\ and\ (5, 12)}}$.

I-4. The d^{th} positive multiple of 999 can be written as $999d = 1000d - d$, which is an integer that is d less than the number $A = 1000d$. For any positive integer d, the 100 integers preceding A have a 9 in the hundreds digit, and the smallest of them is $B = A - 100$. The 10 integers preceding B have a 9 in the tens digit, and the smallest of them is $C = B - 10$. The integer preceding C, which is $C - 1$, has a 9 in the ones digit. Therefore, d must be greater than $1 + 10 + 100 = 111$ to ensure that $999d$ does not have any 9's in its last three digits. If $d = 112$, then there are no 9's at all because $999 \cdot 112 = \boxed{\mathbf{111888}}$.

I-5. Solution 1: Every palindrome $abba$ can be put into one-to-one correspondence with the palindrome $cddc$ where $c = 10 - a$ and $d = 9 - b$. Note that they are different palindromes because $b + d = 9$ implies that $b \neq d$, and also note that the pair of palindromes always add to 11000. There are 90 palindromes (45 pairs) since there are 9 choices for a and 10 choices for b. So, the sum is $45 \cdot 11000 = \boxed{\mathbf{495000}}$.

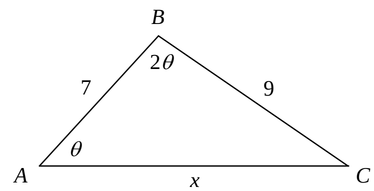
Solution 2: Note that $1 + 2 + \cdots + 99 = 99 \cdot 100/2 = 4950$. In the palindrome $abba$, the number ab formed by the first two digits can equal $10, 11, \ldots, 99$ which sum to $4950 - (1 + 2 + \cdots + 9) = 4950 - 45 = 4905$. The number ba formed by the last two digits can equal any number from 01 through 99 except multiples of 10, and this set of numbers sum to $4950 - (10 + 20 + \cdots + 90) = 4950 - 450 = 4500$. Thus, the total sum of all the palindromes is $4905 \cdot 100 + 4500 = \boxed{\mathbf{495000}}$.

I-6. By Vieta's Theorem, the coefficients and roots of the equation are related as follows:

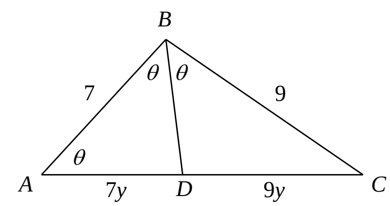
$$\begin{aligned} r_1 + r_2 + r_3 &= r_1 \\ r_1r_2 + r_1r_3 + r_2r_3 &= 2012r_2 \\ r_1r_2r_3 &= r_3 \end{aligned}$$

From the first equation we conclude $r_2 + r_3 = 0$. This implies that $r_1r_2 + r_1r_3 = 0$, so from the second equation we conclude that $r_3 = 2012$ and hence $r_2 = -2012$. From the third equation we conclude that $r_1r_2 = 1$, hence $r_1 = -1/2012$. The answer is $\boxed{(\mathbf{-1/2012}, \mathbf{-2012}, \mathbf{2012})}$.

I-7. Solution 1: Let $AC = x$, and let $\text{m}\angle A = \theta$ so that $\text{m}\angle B = 2\theta$. By the Law of Sines, $\frac{\sin\theta}{9} = \frac{\sin 2\theta}{x} = \frac{2\sin\theta\cos\theta}{x}$, so then $\cos\theta = \frac{x}{18}$. By the Law of Cosines, $9^2 = 7^2 + x^2 - 14x\cos\theta = 49 + x^2 - \frac{7}{9}x^2 = 49 + \frac{2}{9}x^2$. Solving yields $AC = x = \boxed{\mathbf{12}}$.



Solution 2: Let point D on $\overline{AC}$ be the foot of the angle bisector of $\angle ABC$, so $\angle CBD \cong \angle ABD \cong \angle BAC$. The Angle Bisector Theorem gives $AD/CD = 7/9$. Let $CD = 9y$, then $AC = 16y$. Since triangles CBD and CAB are similar, we conclude that $CD/CB = CB/AC \Rightarrow 9y/9 = 9/16y \Rightarrow y^2 = 9/16 \Rightarrow y = 3/4$. Thus, $AC = 16y = \boxed{\mathbf{12}}$.



I-8. Consider the polynomial $g(x) = xf(x) - 1$. Each of the numbers $1, 2, \ldots, 2011$ is a root of $g(x)$. Since the degree of $g(x)$ is 2011,

$$g(x) = xf(x) - 1 = c(x-1)(x-2)\cdots(x-2011)$$

for some constant c. To determine that constant, note that $g(0) = -1 = c(-1)(-2)\cdots(-2011)$, so $c = 1/2011!$. Furthermore, $g(2012) = 2012f(2012) - 1 = \frac{1}{2011!}(2011)(2010)\cdots(1) = 1$. Thus, $f(2012) = 2/2012 = \boxed{\mathbf{1/1006}}$.

I-9. If the prime factorization of n is $n = p_1^{e_1}p_2^{e_2}p_3^{e_3}\cdots p_k^{e_k}$, then n has $(e_1+1)(e_2+1)\cdots(e_k+1)$ divisors. Hence, for n to have a prime number of divisors, n can only have one prime factor p and for n to be less than 100 it must be of the form p, p^2, p^4, or p^6. There are 25 prime numbers less than 100, and $2^2, 3^2, 5^2, 7^2, 2^4, 3^4, 2^6$ are the only other possibilities less than 100, for a total of $\boxed{\mathbf{32}}$. (Note: by definition 1 is not a prime number!)

I-10. From the wording of the problem, one can implicitly assume that the sum converges, but it can be proven directly from the definition of the sequence. To do so, we first note that $a_n > 0$ and $\frac{1}{10}a_n \le a_{n+1} \le a_n$ for all integers $n \ge 1$. The latter inequality can be proven by induction as follows. Note that the inequality holds true for $n = 1$. If it also holds true for $n = k - 1$ for some $k \ge 2$ then we can state:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{20}a_{k-1} + \tfrac{1}{12}a_k \le \tfrac{1}{2}a_k + \tfrac{1}{12}a_k = \tfrac{7}{12}a_k \le a_k \\ a_{k+1} &= \tfrac{1}{20}a_{k-1} + \tfrac{1}{12}a_k \ge \tfrac{1}{20}a_k + \tfrac{1}{12}a_k = \tfrac{2}{15}a_k \ge \tfrac{1}{10}a_k \end{aligned}$$

Hence, $\frac{1}{10}a_k \le a_{k+1} \le a_k$, thus completing the induction.

Returning to the proof of convergence, note that $0 < a_n = \frac{1}{20}a_{n-2} + \frac{1}{12}a_{n-1} \leq \frac{1}{2}a_{n-1} + \frac{1}{12}a_{n-1} = \frac{7}{12}a_{n-1}$ for integers $n \geq 3$. So, $0 < a_n \leq \left(\frac{7}{12}\right)^{n-2} a_2 = \left(\frac{7}{12}\right)^{n-2}$ for integers $n \geq 3$, and since a geometric series with common ratio of $7/12$ converges, the desired infinite sum also converges.

Solution 1: Let the desired sum be $X = a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$. Since all elements of the sum are positive, we can rearrange them in any order:

$$
\begin{aligned}
X &= \sum_{k=1}^{\infty} a_k \\
&= a_1 + a_2 + \sum_{k=3}^{\infty} a_k \\
&= a_1 + a_2 + \sum_{k=3}^{\infty} \left(\frac{1}{20} a_{k-2} + \frac{1}{12} a_{k-1}\right) \\
&= a_1 + a_2 + \sum_{k=1}^{\infty} \frac{1}{20} a_k + \sum_{k=2}^{\infty} \frac{1}{12} a_k \\
&= a_1 + a_2 + \frac{1}{20} X + \frac{1}{12}(X - a_1) \\
&= \frac{23}{12} + \frac{2}{15} X
\end{aligned}
$$

Solving yields $X = \boxed{\mathbf{115/52}}$.

Solution 2: Consider $a_2 = 1$. This term contributes $(1)\left(\frac{1}{12}\right)$ to a_3 and $(1)\left(\frac{1}{20}\right)$ to a_4. In turn, these terms respectively contribute $(1)\left(\frac{1}{12}\right)\left(\frac{1}{12}\right)$ to a_4 and $(1)\left(\frac{1}{12}\right)\left(\frac{1}{20}\right)$ to a_5, and $(1)\left(\frac{1}{20}\right)\left(\frac{1}{12}\right)$ to a_5 and $(1)\left(\frac{1}{20}\right)\left(\frac{1}{20}\right)$ to a_6. Continuing, every term reappears multiplied by $\frac{1}{12}$ or $\frac{1}{20}$ depending on whether we look forward one or two terms down the sequence, respectively. Ultimately, for every integer $k \geq 0$ and $m = 0, 1, \ldots, k$, the product of every permutation of m copies of $\frac{1}{12}$ and $k - m$ copies of $\frac{1}{20}$ will contribute to some term in the sequence. For any given k, the binomial expansion of $\left(\frac{1}{20} + \frac{1}{12}\right)^k$ exactly contains all such permutations, so the total contribution of all the terms resulting from $a_2 = 1$ is given by the infinite series:

$$
\begin{aligned}
&1 + \left(\tfrac{1}{20} + \tfrac{1}{12}\right) + \left(\tfrac{1}{20} + \tfrac{1}{12}\right)^2 + \left(\tfrac{1}{20} + \tfrac{1}{12}\right)^3 + \cdots + \left(\tfrac{1}{20} + \tfrac{1}{12}\right)^k + \cdots \\
&= 1 + \tfrac{2}{15} + \left(\tfrac{2}{15}\right)^2 + \left(\tfrac{2}{15}\right)^3 + \cdots \\
&= \frac{1}{1 - \frac{2}{15}} \\
&= \frac{15}{13}
\end{aligned}
$$

The terms emanating from the term $a_1/20$ that appears in a_3 are the same, but all multiplied by $1/20$, and therefore contribute $(1/20)(15/13) = 3/52$ to the total sum. The only other term not accounted for is $a_1 = 1$. Thus, the total desired sum is $1 + 3/52 + 15/13 = \boxed{\mathbf{115/52}}$.

Team Round Solutions

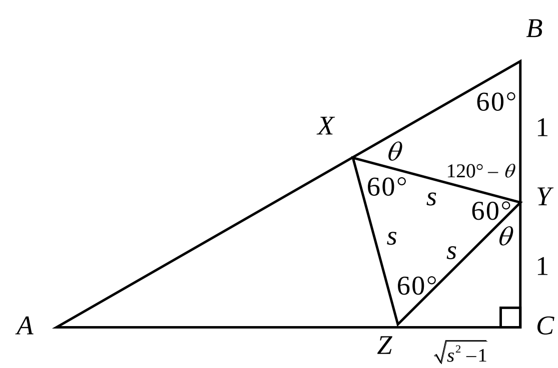
T-1. If one age has one digit then the other age has three digits, and in 31 years the concatenation will not have exactly four digits. Since an age must have at least one digit, both present ages must have exactly two digits, and 31 years from now, both ages will still have two digits. Now, let p^2 be the four-digit number formed by Anne and Tom's present ages and let q^2 be the four-digit number formed by their ages in 31 years. Then, $q^2 - p^2 = 3131$, which can be factored as $(q+p)(q-$ $p=10131$. Since p and q are two-digit positive integers, $q+p>q-p$, so it must be that $q+p=101$ and $q-p=31$ (and not $q+p=3131$ and $q-p=1$). Subtract the two equations to get $2p=70$, so $p=35$, and $p^2 = \boxed{\mathbf{1225}}$. (Indeed, in 31 years the 4-digit number formed by their ages is $q^2 = 66^2 = 4356$.)

T-2. The two equations imply that $3a^2+8a+6=2b^2+7b+2$ (*) and $a^2+4a+6=b^2+2$ (**). Rearrange (**) to get $a^2+4a+4=b^2 \Rightarrow (a+2)^2=b^2$, and since both a and b are positive integers, $a+2=b$. One can substitute this equation in (*) and solve for a, but to make the algebra simpler, first subtract two times equation (**) from (*) to yield $a^2-6=7b-2$, then substitute to get $a^2-6=7(a+2)-2$ leading to $0=a^2-7a-18=(a-9)(a+2)$. Therefore, $a=9$ and $b=11$, and the answer is $\boxed{(\mathbf{9},\mathbf{11})}$.

T-3. If the quadratics have integer roots then the discriminants b^2-4c and b^2-4c-4 are perfect squares. The only perfect squares that differ by 4 are 0 and 4 (since if for integers $q \geq p \geq 0$ we have $4=q^2-p^2=(q+p)(q-p)$, q and p must have the same parity, so $q-p=2$ and $q+p=2$, whose solution yields $(p^2,q^2)=(0,4)$). Thus, if there exists an integer c with the desired property then $b^2-4c=4 \Rightarrow b=2\sqrt{c+1}$, implying that $c+1$ is a perfect square and b is even. Furthermore, if the desired c exists, then the roots of x^2+bx+c are $x=-\frac{b}{2}\pm 1$ and $x^2+bx+c+1$ has a double root at $x=-\frac{b}{2}$, and all of these roots are integers since b is even. Since $44^2=1936$ and $45^2=2025$, the only integer solutions for c are $1^2-1, 2^2-1, \ldots, 44^2-1$, and these correspond to $b=2,4,\ldots,88$. So, the total number of values for b with the desired property is $\boxed{\mathbf{44}}$.

T-4. Note that $a>0$, so then $b>c$. Likewise, $e \geq 0$, so then $d \geq c$. For each choice of (b,c,d) such that $b>c$ and $d \geq c$, there is a unique pair of digits (a,e) such that $abcde$ has the desired property. For each value of c, there are thus $9-c$ choices for b and $10-c$ choices for d. Thus, the answer is $\sum_{c=0}^{9}(9-c)(10-c)=90+72+56+42+30+20+12+6+2+0=\boxed{\mathbf{330}}$.

T-5. Let $m\angle ZYC=\theta$ and let $XY=YZ=XZ=s$. Then, $m\angle BYX=120°-\theta$ and $m\angle BXY=\theta$. Note that $\angle C$ is a right angle, so $ZC=\sqrt{s^2-1}$ by the Pythagorean Theorem. Applying the Law of Sines to triangle BXY yields $\frac{\sin 60°}{s}=\frac{\sin\theta}{1}$. From right triangle ZYC we can determine $\sin\theta=\frac{\sqrt{s^2-1}}{s}$, thus $\frac{\sqrt{3}/2}{s}=\frac{\sqrt{s^2-1}}{s}$. Then, $s^2=\frac{7}{4}$, and the area of XYZ is $\frac{s^2\sqrt{3}}{4}=\boxed{\frac{\mathbf{7\sqrt{3}}}{\mathbf{16}}}$.



T-6. One can show that for $n \geq 2$,

$$\sqrt{1+\frac{1}{(n-1)^2}+\frac{1}{n^2}} = \sqrt{\frac{n^2(n-1)^2+(n-1)^2+n^2}{n^2(n-1)^2}}$$
$$= \sqrt{\frac{n^2(n-1)^2+2n^2-2n+1}{n^2(n-1)^2}}$$
$$= \sqrt{\frac{n^2(n-1)^2+2n(n-1)+1}{n^2(n-1)^2}}$$
$$= \sqrt{\frac{(n(n-1)+1)^2}{n^2(n-1)^2}}$$
$$= 1+\frac{1}{n(n-1)}$$

Hence, the desired sum is:

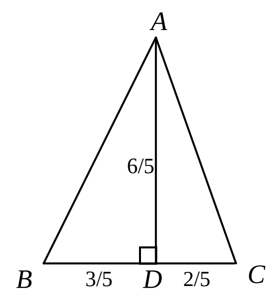
$$\left(1+\tfrac{1}{1\cdot 2}\right)+\left(1+\tfrac{1}{2\cdot 3}\right)+\cdots+\left(1+\tfrac{1}{2011\cdot 2012}\right) = 2011+\left(\tfrac{1}{1\cdot 2}+\tfrac{1}{2\cdot 3}+\cdots+\tfrac{1}{2011\cdot 2012}\right)$$

Furthermore, the identity $\frac{1}{(n-1)n}=\frac{1}{n-1}-\frac{1}{n}$ holds for $n \geq 2$. Therefore, the sum telescopes to

$$2011+\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2011\cdot 2012}\right) = \left(\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2011}-\frac{1}{2012}\right)\right)$$
$$= 2011+\left(1-\tfrac{1}{2012}\right)$$
$$= 2012-\tfrac{1}{2012}$$

The final answer is $(2012^2-1)/2012 = \boxed{\mathbf{4048143/2012}}$. (The answer in the form $\boxed{\mathbf{2011\tfrac{2011}{2012}}}$ is also acceptable).

T-7. <u>Solution 1:</u> Let the triangle be ABC with A the smallest angle and C the largest angle. Since $\mathrm{m}\angle A \leq 60°$, $\tan A \leq \sqrt{3}$, and since $\tan A$ is an integer it must be 1. Note also that no angle is a right angle because all tangents are finite. Thus, we can apply the Law of Tangents: $\tan A+\tan B+\tan C=\tan A\tan B\tan C$. Since $\tan C \neq 1$ (otherwise, $\mathrm{m}\angle A=\mathrm{m}\angle C=45°$ so then $\angle B$ is a right angle, which is impossible), we can solve as follows:



$$\tan B=\tfrac{\tan C+1}{\tan C-1}=1+\tfrac{2}{\tan C-1}$$

For $\tan B$ to be an integer, $\tan C-1$ must be an integer divisor (positive or negative) of 2, so $\tan C-1$ equals 2, 1, –1, or –2. Thus, the only solutions are, respectively, $(\tan B, \tan C)=(2,3),(3,2),(-1,\ 0),(0,-1)$. Since B and C are angles whose measures are between 0° and 180°, and C is the largest angle, the only solution is $(\tan B, \tan C)=(2,3)$.

Now, let $\overline{AD}$ be the altitude to $\overline{BC}$. Since A, B, and C are angles of a triangle whose tangents are all positive, they are all acute angles, and D is between B and C (see diagram). Dividing $\tan B=$

$AD/BD = 2$ by $\tan C = AD/CD = 3$ yields $CD/BD = 2/3$. Also, note that $BD + CD = BC = 1$. Solving these equations simultaneously yields the values $BD = 3/5$ and $CD = 2/5$. Then, $AD = (3/5)\tan B = 6/5$, and by the Pythagorean Theorem, the longest side $AB = \boxed{\mathbf{3\sqrt{5}/5}}$.

Solution 2: Determine the values of $\tan A$, $\tan B$, and $\tan C$ as in Solution 1. Since $\tan A = 1$, A is an acute angle, so $\sin A = \sqrt{2}/2$. Since $\tan C = 3$, C is an acute angle, so $\sin C = \frac{\tan C}{\sqrt{1+\tan^2 C}} = \frac{3\sqrt{10}}{10}$. The longest side $\overline{AB}$ and the shortest side $\overline{BC}$ are opposite to the largest angle C and the smallest angle A, respectively. By the Law of Sines, $AB/\sin C = BC/\sin A$, so $AB = BC(\sin C/\sin A) = (1)\left(3\sqrt{10}/10\right)/\left(\sqrt{2}/2\right) = \boxed{\mathbf{3\sqrt{5}/5}}$.

T-8. Solution 1: We wish to determine the value of

$$\begin{aligned}\tan 20^\circ \tan 40^\circ \tan 80^\circ &= \frac{\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ} \\ &= \frac{(2\sin 10^\circ \cos 10^\circ)(2\sin 20^\circ \cos 20^\circ)(2\sin 40^\circ \cos 40^\circ)}{\cos 20^\circ \cos 40^\circ \cos 80^\circ} \\ &= 8\cos 10^\circ \sin 20^\circ \sin 40^\circ\end{aligned}$$

where in the last line we used the fact that $\sin 10^\circ = \cos 80^\circ$. Continuing, we note the following identities: $\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(y-x))$ and $\sin x \sin y = \frac{1}{2}(\cos(y-x) - \cos(x+y))$. Therefore,

$$\begin{aligned}8\cos 10^\circ \sin 20^\circ \sin 40^\circ &= 4(\sin 30^\circ + \sin 10^\circ)\sin 40^\circ \\ &= 2\sin 40^\circ + 4\sin 10^\circ \sin 40^\circ \\ &= 2\sin 40^\circ + 2(\cos 30^\circ - \cos 50^\circ) \\ &= 2\cos 30^\circ\end{aligned}$$

noting that $\sin 30^\circ = 1/2$ and $\sin 40^\circ = \cos 50^\circ$. Hence, the desired answer is $\boxed{\mathbf{\sqrt{3}}}$.

Solution 2: The product of the tangent addition and subtraction identities yields

$$\begin{aligned}\tan(a+b)\tan(a-b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b}\frac{\tan a - \tan b}{1 + \tan a \tan b} \\ &= \frac{\tan^2 a - \tan^2 b}{1 - \tan^2 a \tan^2 b}\end{aligned}$$

Thus,

$$\begin{aligned}\tan 20^\circ \tan(60^\circ - 20^\circ)\tan(60^\circ + 20^\circ) &= \tan 20^\circ \frac{\tan^2 60^\circ - \tan^2 20^\circ}{1 - \tan^2 60^\circ \tan^2 20^\circ} \\ &= \frac{3\tan 20^\circ - \tan^3 20^\circ}{1 - 3\tan^2 20^\circ}\end{aligned}$$

Since the tangent triple angle identity is $\tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$, the above expression must equal $\tan(3 \cdot 20^\circ) = \tan 60^\circ = \boxed{\mathbf{\sqrt{3}}}$.

T-9. Note that $\left(n+\frac{1}{2}\right)^2 = n^2+n+\frac{1}{4}$. Therefore, for integers $n \geq 1$, $[\![x]\!] = n$ only for those integers x satisfying $(n-1)^2+(n-1)+1 \leq x \leq n^2+n$, a range that contains $2n+2$ integers. Since $n=99$ yields $n^2+n=9900$, the desired sum equals:

$$\begin{aligned}100+\sum_{n=1}^{99}(2n+2)(n) &= 100+2\sum_{n=1}^{99}(n^2+n)\\ &= 100+2\left(\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right)_{n=99}\\ &= 100+2\left(\frac{99(100)(199)}{6}+\frac{99(100)}{2}\right)\end{aligned}$$

The final expression simplifies to $\boxed{\mathbf{666700}}$.

T-10. Clearly, the shortest representation of 1111 is a sequence of $n=4$ ones with no plus signs. For a sequence of $n>4$ ones, 1111 only results if 1, 11, and 111 are the only numbers formed by the plus signs. Suppose these numbers appear a, b, and c times in the sum, respectively. Then, $a+2b+3c=n$ (*) and $a+11b+111c=1111$ (**). Taking (**) modulo 9 yields $a+2b+3c \equiv 4 \mod 9$, so $n \equiv 4 \mod 9$ also.

It is possible to form 1111 from $n=31$ ones, using one copy of 1 and ten copies of 111 (i.e., $a=1$, $b=0$, $c=10$). It is not possible for 1111 to result from a sequence of $n=13, 22$, or 40 ones. To prove this, we first note that a, b, and c are non-negative integers, hence, $2b=n-(a+3c) \leq n$. Then, subtracting (*) from (**) yields a lower bound for c as follows: $9b+108c=1111-n \Rightarrow 108c=1111-n-9b \geq 1111-n-9n/2 \Rightarrow c \geq (1111-11n/2)/108$. From (**) we obtain an upper bound for c as follows: $111c=1111-(a+11b) \leq 1111 \Rightarrow c \leq 10$. Therefore:

- If $n=13$, $c \geq (2079/2)/108 = 77/8 > 9$. Thus, $c=10$, implying $n \geq 30$, a contradiction.
- If $n=22$, $c \geq 990/108 = 55/6 > 9$. Thus, $c=10$, implying $n \geq 30$, a contradiction.
- If $n=40$, $c \geq 891/108 = 33/4 > 8$. Thus, $c=9$ or 10. If $c=9$, then (*) and (**) simplify to $a+2b=13$ and $a+11b=112$ whose simultaneous solution is $b=11$ and $a=-9$, which is impossible since a is non-negative. Similarly, if $c=10$, then $a+2b=10$ and $a+11b=1$ so then $a=12$ and $b=-1$, which is also impossible.

We have covered all values of $n \equiv 4 \mod 9$ such that $n<49$. Now, we claim that 1111 can be obtained for every $n \equiv 4 \mod 9$ where $49 \leq n \leq 1111$. To prove this, we begin with the representation of 1111 discussed above that uses one 1 and ten copies of 111. By successively replacing one copy of 111 with one 1 and ten copies of 11, we get representations of 1111 using $31+18k$ ones, for $k=1,2,\dots,10$. Each of these representations includes at least one copy of 11. To get representations of $31+9p$ ones for odd values of $p=3,5,\dots,19$, take the representation using $31+18((p-1)/2)=31+9(p-1)$ ones and replace one of the copies of 11 with eleven ones. Since $31+18(10)=211$, we have so far covered the range $49 \leq n \leq 211$. For the rest, note that the above representation with 211 ones has 100 copies of 11 and 11 copies of 1. Successively replacing each 11 with eleven ones generates 9 more ones each time, until the sum consists of 1111 copies of 1, for a total of $n=1111$ ones. This completes the proof of the claim.

Finally, there are no other valid values of n because $n=a+2b+3c \leq a+11b+111c=1111$. Since there are $(1111-49)/9+1=119$ numbers from 49 to 1111 that are congruent to 4 modulo 9, and by also counting $n=4$ and 31, the answer is $\boxed{\mathbf{121}}$.

Relay Round Solutions

R1-1. Clearly, $k!$ evenly divides $(2012!)!$ if k is an integer within $0 \le k \le 2012!$. Since $6! = 720$, $(((n!)!)!$ evenly divides $(2012!)!$ for $n = 1, 2, \dots, 6$. If $n \ge 7$, then $(((n!)!)! \ge (((7!)!)! = (5040!)! > (2012!)!$ so it is not possible for $(((n!)!)!$ to evenly divide $(2012!)!$. Hence, the answer is $\boxed{6}$.

R1-2. <u>Solution 1:</u> Let the product be P. Then $P = T \cdot \sqrt{T} \cdot \sqrt{\sqrt{T}} \cdot \sqrt{\sqrt{\sqrt{T}}} \cdots = T \cdot \sqrt{T \cdot \sqrt{T} \cdot \sqrt{\sqrt{T}} \cdots} = T\sqrt{P}$. Hence, $P = T^2 = 6^2 = \boxed{\mathbf{36}}$.

<u>Solution 2:</u> $T \cdot \sqrt{T} \cdot \sqrt{\sqrt{T}} \cdot \sqrt{\sqrt{\sqrt{T}}} \cdots = T^1 \cdot T^{1/2} \cdot T^{1/4} \cdot T^{1/8} \cdots = T^{\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)} = T^2 = 6^2 = \boxed{\mathbf{36}}$.

R1-3. The second term of an arithmetic sequence is the average of the first and third term. Applying this property to the rightmost column implies the square in the lower right corner has value -1988. Applying it again to the bottom row implies the middle square of that row has value $\frac{1}{2}(T - 1988) = \frac{1}{2}T - 994$. Applying it once more to the middle column yields $x = \frac{1}{2}\left(\frac{1}{2}T - 994 + 20\right) = \frac{1}{4}T - 487 = \frac{1}{4}(36) - 487 = \boxed{\mathbf{-478}}$.

-1972	20	2012
$\frac{1}{2}T - 986$	$\frac{1}{4}T - 487$	12
T	$\frac{1}{2}T - 994$	-1988

(Similarly, one can determine the values of the empty squares in the leftmost column, and arrive at the same answer for x. The resulting 3×3 grid is shown above with squares filled out in terms of T.)

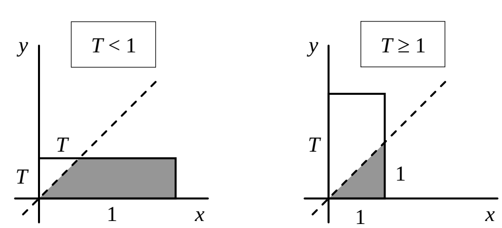
R2-1. For the expression to equal one, either (i) the exponent is $x^2 + 3x + 2 = 0$ for $x \neq 3$, or (ii) the base is $x - 3 = 1$, or (iii) the base is $x - 3 = -1$ and the exponent is even. In the first case, $x^2 + 3x + 2 = (x + 2)(x + 1) = 0$ so $x = -2$ or -1. In the second case, $x = 4$, and in the third case, $x = 2$ (and the exponent is 12). So the answer is $\boxed{\mathbf{-2}}$.

R2-2. Note that

$$\begin{aligned} f\left(f\left(f(x)\right)\right) &= f\left(f(ax + b)\right) \\ &= f\left(a(ax + b) + b\right) \\ &= a(a(ax + b) + b)) + b \\ &= a^3x + a^2b + ab + b \end{aligned}$$

Therefore, $f(f(f(1))) - f(f(f(0))) = a^3 + (a^2b + ab + b) - (a^2b + ab + b) = a^3 = 25 - T = 25 - (-2) = 27$, making $a = \boxed{3}$.

R2-3. Consider the rectangle with corners at $(0, 0)$, $(1, 0)$, $(0, T)$, and $(1, T)$. The desired probability is the fraction of the rectangle below the line $y = x$. If $T < 1$, it is $\frac{T - (T^2/2)}{T} = 1 - \frac{T}{2}$. If $T \ge 1$, it is $\frac{1/2}{T} = \frac{1}{2T}$. Since $T = 3$, the answer is $\frac{1}{2 \cdot 3} = \boxed{\frac{1}{6}}$.



Power Round Solutions

1a. For all integers x, the integers x and $x+1$ are of opposite parity, and one of them is evenly divisible by two. Hence, $f(x) = x(x+1)/2$ is integer-valued.

1b. The non-constant polynomial $g(x) = x + \frac{1}{2}$ has rational coefficients and is never an integer for all integers x.

2. For $n = 0$, we have $\Delta p_1(x) = (x+1) - (x) = 1 = p_0(x)$. For $n \geq 1$, we have:

$$\begin{aligned}
\Delta p_{n+1}(x) &= p_{n+1}(x+1) - p_{n+1}(x) \\
&= \frac{(x+1)(x)\cdots\big(x+1-(n-1)\big)(x+1-n)}{(n+1)(n)(n-1)\cdots(1)} - \frac{(x)(x-1)\cdots\big(x-(n-1)\big)(x-n)}{(n+1)(n)(n-1)\cdots(1)} \\
&= \frac{(x)(x-1)\cdots\big(x-(n-1)\big)}{(n+1)(n)(n-1)\cdots(1)}\big((x+1)-(x-n)\big) \\
&= \frac{(x)(x-1)\cdots\big(x-(n-1)\big)}{(n+1)(n)(n-1)\cdots(1)}(n+1) \\
&= \frac{(x)(x-1)\cdots\big(x-(n-1)\big)}{(n)(n-1)\cdots(1)} \\
&= p_n(x)
\end{aligned}$$

3a. If $n = 0$, then $p_0(x) = 1$, which is never zero. If $n > 0$, by direct substitution, the n distinct values $x = 0, 1, \dots, n-1$ yield $p_n(x) = 0$. Since p_n is a polynomial of degree n, it has at most n distinct roots (i.e. values of x for which $p_n(x) = 0$), and there are no other solutions. So, the desired x are:

$$x = \begin{cases} \text{no solutions}, & n = 0 \\ 0, 1, \dots, n-1, & n > 0 \end{cases}$$

3b. If $n = 0$, then $p_0(x) = 1$, which is always an integer for all integers x. If $n > 0$, then for integers $x \geq n$, we have $p_n(x) = \binom{x}{n}$, which is always an integer because it is the number of ways to choose subsets of n elements from a set of x distinct elements, where the order of the n elements does not matter. For integers $0 \leq x < n$, from Problem 3a, the result is a specific integer, namely $p_n(x) = 0$. For integers $x < 0$, we can rewrite $p_n(x)$ as

$$\begin{aligned}
p_n(x) &= \frac{(x)(x-1)(x-2)\cdots\big(x-(n-1)\big)}{(n)(n-1)\cdots(1)} \\
&= \frac{(-1)^n}{n!}(-x)(-x+1)(-x+2)\cdots(-x+n-1) \\
&= \frac{(-1)^n}{n!}(-x+n-1)\big((-x+n-1)-1\big)\cdots\big((-x+n-1)-(n-1)\big) \\
&= (-1)^n p_n(-x+n-1)
\end{aligned}$$

which is an integer because $-x+n-1 = n-(x+1)$ is an integer greater than or equal to n. Thus, $p_n(x)$ is an integer for all integers x.

4a. If f is integer-valued then $f(x+1)$ and $f(x)$ are integers for all integers x. Hence, their difference $\Delta f(x) = f(x+1) - f(x)$ is also an integer for all integers x. This proves that if f is any integer-valued function, then Δf is also integer-valued.

4b. Use the definition of Δ as follows:

$$\begin{aligned}\Delta h(x) &= h(x+1) - h(x)\\ &= \big(uf(x+1) + vg(x+1)\big) - \big(uf(x) + vg(x)\big)\\ &= u\big(f(x+1) - f(x)\big) + v\big(g(x+1) - g(x)\big)\\ &= u\Delta f(x) + v\Delta g(x)\end{aligned}$$

5. Any polynomial f of degree $n \geq 1$ can be written as $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ where $c_n \neq 0$. Then, Δf is a polynomial of the form:

$$\begin{aligned}\Delta f(x) &= f(x+1) - f(x)\\ &= (c_n(x+1)^n + c_{n-1}(x+1)^{n-1} + \cdots + c_0) - (c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0)\end{aligned}$$

The highest term in the parentheses on the left is $c_n x^n$, from the expansion of $c_n(x+1)^n$. The highest term in the parentheses on the right is also $c_n x^n$. So these terms cancel out.

The second-highest term in the parentheses on the left is $(nc_n + c_{n-1})x^{n-1}$ which comes from the expansion of $c_n(x+1)^n$ and $c_{n-1}(x+1)^{n-1}$, and the second-highest term in the parentheses on the right is $c_{n-1}x^{n-1}$. Therefore, the subtraction leaves the term $nc_n x^{n-1}$, which has a non-zero coefficient. Hence, Δf is a polynomial whose degree is $n-1$.

6a. Clearly a constant polynomial f has the property that Δf is identically zero. We will show that no other polynomials have this property.

Suppose that f is a polynomial of degree $n = 1$. Then $f(x) = a_1 x + a_0$ for some non-zero a_1, and $\Delta f(x) = (a_1(x+1) + a_0) - (a_1 x + a_0) = a_1$ cannot be identically zero. Now assume that f is a polynomial of degree $n > 1$. From Problem 5 we know Δf is a polynomial with degree $n - 1 > 0$, and thus is non-constant and Δf cannot be identically zero. Therefore, there are no non-constant polynomials f having the property that Δf is identically zero.

6b. One example is $g(x) = \sin(2\pi x)$ which is a sinusoid with period 1. This continuous function is not a polynomial because it is not identically zero (e.g., $g(1/4) = 1$) and it has infinitely many roots ($g(n) = 0$ for all integers n). Furthermore, $\Delta g(x) = \sin\big(2\pi(x+1)\big) - \sin(2\pi x) = \sin(2\pi x + 2\pi) - \sin(2\pi x) = \sin(2\pi x) - \sin(2\pi x) = 0$, for all x.

7. Define the polynomial h as $h(x) = f(x) - g(x)$ for all x. So, by Problem 4b, $\Delta h(x) = \Delta f(x) - \Delta g(x)$ for all x. Now, if for all x, $\Delta f(x) = \Delta g(x)$, then $\Delta f(x) - \Delta g(x) = 0 = \Delta h(x)$ so Problem 6a implies that $h(x) = f(x) - g(x) = C$, or equivalently $f(x) = g(x) + C$, for some constant C. This equation also implies that if $g(x)$ is an integer for all integers x, then C must be an integer for $f(x)$ to also be an integer for all integers x.

8. In this solution, we will say that the polynomial f is an *integer linear combination* of $p_n, p_{n-1}, \dots, p_0$ if there exist integer coefficients $a_n, a_{n-1}, \dots, a_0$ such that $f(x) = a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x)$ for all x.

First, we wish to prove the forward statement, which is equivalent to the statement that, for all integers $n \geq 0$, all polynomials f of degree n that are integer linear combinations of $p_n, p_{n-1}, \dots, p_0$ are integer-valued. This is straightforward, because by Problem 3b, $p_n(x)$ is an integer for all integers x and all integers $n \geq 0$. Therefore, for all integers $n \geq 0$, all polynomials f of degree n that are integer linear combinations of $p_n, p_{n-1}, \dots, p_0$ have the property, for all integers x, that $f(x)$ equals the sum of products of integers, which always results in an integer.

Second, we wish the prove the reverse statement, which is equivalent to the statement that, for all integers $n \geq 0$, all integer-valued polynomials f of degree n are integer linear combinations of $p_n, p_{n-1}, \dots, p_0$. We will prove this by induction. For $n = 0$, we consider all integer-valued polynomials f of degree zero. All such f are constant integers and can be written as an integer times $p_0(x) = 1$. Thus, all polynomials f of degree zero are integer linear combinations of p_0.

Now assume, for some integer $k > 0$, that all integer-valued polynomials of degree k are integer linear combinations of $p_k, p_{k-1}, \dots, p_0$. Consider any given integer-valued polynomial f of degree $k + 1$. Then, by Problem 5, Δf is an integer-valued polynomial of degree k, and for all x, can be written as

$$\Delta f(x) = a_k p_k(x) + a_{k-1} p_{k-1}(x) + \cdots + a_0 p_0(x)$$

for some integers $a_k, a_{k-1}, \dots, a_0$. Using Problem 2, we can then write

$$\Delta f(x) = a_k \Delta p_{k+1}(x) + a_{k-1} \Delta p_k(x) + \cdots + a_0 \Delta p_1(x)$$

for all x. It is easy to show by induction that the distributive property of Δ (Problem 4b) generalizes to any number of addends, therefore $\Delta f(x) = \Delta h(x)$ for all x where

$$h(x) = a_k p_{k+1}(x) + a_{k-1} p_k(x) + \cdots + a_0 p_1(x)$$

Problem 7 then implies that, for all x,

$$f(x) = h(x) + C = \big(a_k p_{k+1}(x) + a_{k-1} p_k(x) + \cdots + a_0 p_1(x)\big) + C$$

for some integer C. Therefore, f satisfies, for all x,

$$f(x) = a_k p_{k+1}(x) + a_{k-1} p_k(x) + \cdots + a_0 p_1(x) + C p_0(x)$$

All of the coefficients in the above expression are integers, and this is true of any integer-valued polynomial f of degree $k + 1$. Hence, all integer-valued polynomials of degree $k + 1$ are integer linear combinations of $p_{k+1}, p_k, \dots, p_0$ thus completing the induction.

9. Solution 1: We have $f(x) = a_n p_n(x) + a_{n-1} p_{n-1}(x) + \cdots + a_0 p_0(x) = b_n p_n(x) + b_{n-1} p_{n-1}(x) + \cdots + b_0 p_0(x)$ for all x. Two polynomials of x are equal for all x only if the coefficients of each power of x are the same. Since only $p_n(x)$ has the term x^n, this implies that $a_n x^n / n! = b_n x^n / n!$, so $a_n = b_n$. Now consider

$$f(x) - a_n p_n(x) = a_{n-1} p_{n-1}(x) + \cdots + a_0 p_0(x) = b_{n-1} p_{n-1}(x) + \cdots + b_0 p_0(x)$$

Again, the coefficients of each power of x must be the same. Since only $p_{n-1}(x)$ has the term x^{n-1}, this implies that $a_{n-1} x^{n-1}/(n-1)! = b_{n-1} x^{n-1}/(n-1)!$, so $a_{n-1} = b_{n-1}$. By repeating this reasoning, we can conclude that $a_i = b_i$ for $i = 0, 1, \dots, n$.

Solution 2: Note that $p_n(x) = 0$ for $x = 0, 1, \dots, n-1$ for all integers $n \geq 1$ (Problem 3a) and $p_n(n) = 1$ for all integers $n \geq 0$.

We have $a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x) = b_n p_n(x) + b_{n-1}p_{n-1}(x) + \cdots + b_0 p_0(x)$ for all x. By taking $x = 0$, remembering that $p_n(0) = 0$ for all integers $n > 0$ and $p_0(0) = 1$, the equation simplifies to $a_0 = b_0$. So, $a_0 p_0(x) = b_0 p_0(x)$ for all x. If $n > 0$, take $x = 1$, remembering that $p_n(1) = 0$ for all integers $n > 1$ and $p_1(1) = 1$, so then the equation simplifies to $a_1 + a_0 p_0(1) = b_1 + b_0 p_0(1) \Rightarrow a_1 = b_1$. Next, if $n > 1$, take $x = 2$, remembering that $p_n(2) = 0$ for all integers $n > 2$ and $p_2(2) = 1$, so then the equation simplifies to $a_2 + a_1 p_1(2) + a_0 p_0(2) = b_2 + b_1 p_1(2) + b_0 p_0(2) \Rightarrow a_2 = b_2$. By repeating this reasoning we can conclude that $a_i = b_i$ for $i = 0, 1, \dots, n$.

10a. Algorithm 1: Let f be any integer-valued polynomial of degree n. The algorithm is as follows:

1. Set the value of k equal to n.
2. Represent $f(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$. (On each iteration of this algorithm, the values of the c's can be different.)
3. Choose $a_k = k!\, c_k$.
4. Replace $f(x)$ with $f(x) - a_k p_k(x)$, then set the value of k equal to $k - 1$.
5. Repeat steps 2 through 4 if $k \geq 0$, otherwise terminate.

Proof: The proof is conceptually similar to Solution 1 of Problem 9. We wish to write the polynomial $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ as $f(x) = a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x)$. These two expressions are equal only if the coefficients of each power of x are the same. After Step 1, $k = n$, and we are concerned with x^k which in the latter expression is only present in $p_k(x)$. Thus, equating the coefficients of x^k yields $c_k = a_k/k!$ leading to $a_k = k!\, c_k$ given in Step 3.

Continuing with Step 4, still noting that $k = n$, $f(x) - a_k p_k(x)$ is a polynomial of degree $d \leq k - 1$. Furthermore, $f(x)$ and $f(x) - a_k p_k(x)$ have the same integer coefficients a_i for $i = 0, 1, \dots, k-1$ (where, to account for the scenario in which $d < k - 1$, we take a_j to be 0 for $d < j \leq k - 1$). Thus, if we apply Steps 2 and 3 to $f(x) - a_k p_k(x)$ using $k - 1 = n - 1$ instead of k, we can determine the value of a_{n-1}. Likewise, subsequent iterations will then determine $a_{n-2}, a_{n-3}, \dots, a_0$.

The algorithm terminates after determining the last integer coefficient, a_0 (i.e., $k = -1$ at Step 5).

Algorithm 2: Let f be any integer-valued polynomial of degree n to be written as $f(x) = a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x)$. The algorithm is to compute the integer coefficients using the equation $a_m = f(m) - \sum_{i=0}^{m-1} a_i p_i(m)$ sequentially in the order $m = 0, 1, \dots, n$.

Proof: The proof is conceptually similar to Solution 2 of Problem 9. Note that $p_n(x) = 0$ for $x = 0, 1, \dots, n-1$ for all integers $n \geq 1$ (Problem 3a), and $p_n(n) = 1$ for all integers $n \geq 0$. Thus,

$$\begin{aligned} f(0) &= a_n p_n(0) + a_{n-1}p_{n-1}(0) + \cdots + a_1 p_1(0) + a_0 p_0(0) \\ &= a_0 \end{aligned}$$

$$\begin{aligned} f(1) &= a_n p_n(1) + a_{n-1}p_{n-1}(1) + \cdots + a_1 p_1(1) + a_0 p_0(1) \\ &= a_1 + a_0 p_0(1) \end{aligned}$$

$$\begin{aligned} f(2) &= a_n p_n(2) + a_{n-1}p_{n-1}(2) + \cdots + a_1 p_1(2) + a_0 p_0(2) \\ &= a_2 + a_1 p_1(2) + a_0 p_0(2) \end{aligned}$$

and in general, for $m = 0, 1, \ldots, n$

$$\begin{aligned} f(m) &= a_m + a_{m-1}p_{m-1}(m) + a_{m-2}p_{m-2}(m) + \cdots + a_0p_0(m) \\ &= a_m + \sum_{i=0}^{m-1} a_ip_i(m) \end{aligned}$$

This leads directly to the claimed equation for a_m. Note also that for each m, the computation of a_m depends only on the previously computed values of $a_0, a_1, \ldots, a_{m-1}$, and the values of known polynomials at m. Thus, at each step the equation is a closed formula and the algorithm can be used to sequentially determine the integer coefficients.

<u>Comment:</u> Can you use Algorithm 2 to prove the "reverse" statement of Problem 8 (all integer-valued polynomials f of degree $n \geq 0$ are integer linear combinations of $p_n, p_{n-1}, \ldots, p_0$)?

<u>Comment:</u> What results if either algorithm is applied to a polynomial that is not integer-valued?

10b. <u>Algorithm 1:</u> Note that

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= \tfrac{1}{2}x(x-1) = \tfrac{1}{2}x^2 - \tfrac{1}{2}x \\ p_3(x) &= \tfrac{1}{6}x(x-1)(x-2) = \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x \end{aligned}$$

Clearly f is integer-valued. Thus by Algorithm 1 from the Solution of Problem 10a,

$$\begin{aligned} a_3 &= 2 \cdot 3! = 12 \\ f(x) - a_3p_3(x) &= (2 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 2) - 12\left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x\right) \\ &= 6x^2 - 3x + 2 \to f(x) \end{aligned}$$

Then,

$$\begin{aligned} a_2 &= 6 \cdot 2! = 12 \\ f(x) - a_2p_2(x) &= (6x^2 - 3x + 2) - 12\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x\right) \\ &= 3x + 2 \to f(x) \end{aligned}$$

Then, we can easily see that $a_1 = 3$ and $a_0 = 2$. Overall, $(a_0, a_1, a_2, a_3) = (2, 3, 12, 12)$.

<u>Algorithm 2:</u> Clearly f is integer-valued. Thus, by Algorithm 2 from the Solution of Problem 10a, the integer coefficients are as follows:

$$\begin{aligned} a_0 &= f(0) = 2 \\ a_1 &= f(1) - a_0p_0(0) = 5 - 2 \cdot 1 = 3 \\ a_2 &= f(2) - \big(a_1p_1(2) + a_0p_0(2)\big) = 20 - (3 \cdot 2 + 2 \cdot 1) = 12 \\ a_3 &= f(3) - \big(a_2p_2(3) + a_1p_1(3) + a_0p_0(3)\big) = 59 - (12 \cdot 3 + 3 \cdot 3 + 2 \cdot 1) = 12 \end{aligned}$$

So we obtain the result $(a_0, a_1, a_2, a_3) = (2, 3, 12, 12)$.

Credits

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